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D-spaces and thick covers

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ABSTRACT

The following results are obtained.

– An open neighbornet U of X has a closed discrete kernel if X has an almost thick cover by countably U-close sets.
– Every hereditarily thickly covered space is aD and linearly D.
– Every t-metrizable space is a D-space.
– X is a D-space if X has a cover $\{X_\alpha\colon \alpha < \lambda\}$ by D-subspaces such that, for each $\beta < \lambda$, the set $\bigcup\{X_\alpha\colon \alpha < \beta\}$ is closed.

1. Introduction

The concept of a D-space was introduced by van Douwen, and it first appeared in print in [10]. It is well known that extent coincides with Lindelöf number in a D-space. In particular, every countably compact D-space is compact and every D-space with countable extent is Lindelöf. These results make D-spaces useful in research on covering properties. Interesting work on D-spaces has been done by many topologists, see for example, Arhangel'skii and Buzyakova [3–5,7], Gruenhage [15], Fleissner and Stanley [14], and Liang-Xue Peng [23–28].

In a study of weak topologies of Banach spaces and $C_p(K)$-spaces, Dow, Junnila and Pelant [12] introduced the notion of a thick cover and the classes of thickly covered spaces and t-metrizable spaces. They showed that metaLindelöf spaces are thickly covered, metaLindelöf σ-spaces are t-metrizable, and for every compact Hausdorff space K, the space $C_p(K)$ is t-metrizable.

In Section 3, we consider thickness properties related with D-spaces. We show that an open neighbornet U of X has a closed discrete kernel if X has an "almost thick" cover by countably U-close sets. As a consequence, X is a D-space provided that X has an almost thick cover by closed Lindelöf D-subspaces. We also show that every hereditarily thickly covered space is both aD and linearly D.

In Section 4, we consider t-metrizable spaces and some related spaces. We show that every t-metrizable space has a "predictable" network, and we show that a space with a predictable network is a D-space. As a consequence, every t-metrizable space is a D-space. We indicate some t-metrizable spaces, such as spaces with a point-countably expandable

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network. We show that the results of Section 3 yield as corollaries recent results of Tkachuk and Peng which assert that (weakly) monotonically monolithic spaces are D-spaces. We answer several questions from Tkachuk's paper on monotonically monolithic spaces.

In Section 5, we raise the problem whether X is a D-space provided that X is the union of finitely many subspaces with point-countably expandable networks. Towards a partial solution, we show that X is a D-space provided that X is the union of finitely many subspaces with "strongly point-countably expandable" networks. We also show that X is a D-space if X has a cover $\{X_\alpha\colon \alpha < \lambda\}$ by D-subspaces such that, for each $\beta < \lambda$, the set $\bigcup\{X_\alpha\colon \alpha < \beta\}$ is closed. In particular, X is a D-space if X has a closure-preserving cover by closed D-subspaces.

2. Terminology, notation and a basic lemma

In the following, *space* means a T_1*-space*.

A *neighbornet* (*an open neighbornet*) of a space X is a binary relation U on X such that, for every $x \in X$, the set $U\{x\}$ is a neighborhood (an open neighborhood) of x in X [20].

Let U be a neighbornet of X. A set $A \subset X$ is a *kernel* of U if $U(A) = X$ [8]. For $A \subset F \subset X$, we say that A is a *kernel of* U *in* F if $F \subset U(A)$.

Definition 2.1. ([10]) A space X is a *D-space* provided that every neighbornet of X has a closed discrete kernel.

For a family $\mathcal{L}$ of sets and for a set A, we set $(\mathcal{L})_A = \{L \in \mathcal{L}\colon L \cap A \neq \emptyset\}$; if $A = \{x\}$, then we write $(\mathcal{L})_x$ in room of $(\mathcal{L})_A$. For a set A, we set $[A]^{<\omega} = \{H \subset A\colon |H| < \omega\}$ and $[A]^{\leqslant\omega} = \{H \subset A\colon |H| \leqslant \omega\}$.

Definition 2.2. ([12]) A cover $\mathcal{L}$ of a space X is *thick* if we can assign $\mathcal{L}_H \in [\mathcal{L}]^{<\omega}$ and $L_H = \bigcup \mathcal{L}_H$ to each $H \in [X]^{<\omega}$ in such a way that

$$\overline{A} \subset \bigcup\{L_H\colon H \in [A]^{<\omega}\} \quad \text{for every } A \subset X.$$

The space X is *thickly covered* if every open cover of X is thick.

Note that if a cover $\mathcal{N}$ of X has a thick refinement, then $\mathcal{N}$ is thick.

Definition 2.3. ([12]) A space (X, τ) is *t-metrizable* if there exists a metrizable topology π on X with $\tau \subset \pi$ and an assignment $H \mapsto J_H$ from $[X]^{<\omega}$ to $[X]^{<\omega}$ such that

$$\overline{A}^{\tau} \subset \overline{\bigcup_{H\in[A]^{<\omega}} J_H}^{\pi} \quad \text{for every } A \subset X.$$

Remark. It is enough that the assignment in Definition 2.2 is from $[X]^{<\omega}$ to $[\mathcal{L}]^{\leqslant\omega}$ (see [12, Lemma 2.1]). Also, it is enough that the assignment in Definition 2.3 is from $[X]^{<\omega}$ to $[X]^{\leqslant\omega}$.

A family $\mathcal{L}$ of subsets of a space X is *point-countably expandable* if $\mathcal{L}$ has a *point-countable open expansion*, i.e., there exists a family $\{G_L\colon L \in \mathcal{L}\}$ of open subsets of X such that we have $L \subset G_L$ for every $L \in \mathcal{L}$, and for every $x \in X$, the family $\{L \in \mathcal{L}\colon x \in G_L\}$ is countable.

We refer the reader to [13] and [18] for further definitions of terms used below.

To study D-spaces, we need to construct discrete families associated with neighbornets, and the following simple lemma is useful for that purpose.

Lemma 2.4. *Let V be an open neighbornet of X and $\mathcal{L} = \{L_\alpha\colon \alpha < \lambda\}$ a family of subsets of X such that $L_\alpha \subset X \setminus V(\bigcup_{\beta<\alpha} L_\beta)$ for every $\alpha < \lambda$ and $\overline{\bigcup_{\beta<\gamma} L_\beta} \subset V(\bigcup_{\beta<\gamma} L_\beta)$ for each $\gamma \leqslant \lambda$. Then $\mathcal{L}$ is discrete.*

Proof. Let $x \in X$. We show that x has a neighborhood which meets L_α for at most one α. This is clear if $x \notin \overline{\bigcup_{\beta<\lambda} L_\beta}$. Otherwise, let $\gamma = \min\{\delta \leqslant \lambda\colon x \in \overline{\bigcup_{\beta<\delta} L_\beta}\}$. Since $\overline{\bigcup_{\beta<\gamma} L_\beta} \subset V(\bigcup_{\beta<\gamma} L_\beta)$, there exists $\gamma' < \gamma$ such that $x \in V(L_{\gamma'})$. By minimality of γ, we have that $x \notin \overline{\bigcup_{\beta<\gamma'} L_\beta}$. Now $V(L_{\gamma'}) \setminus \overline{\bigcup_{\beta<\gamma'} L_\beta}$ is a neighborhood of x which does not meet L_α for any $\alpha \neq \gamma'$. $\square$

3. *D*-spaces and thick families

In this section, we show that a neighbornet U of a space X has a closed discrete kernel provided that X has an "almost thick" cover consisting of sets which are "small" with respect to U. To obtain this result, we modify a result of Gruenhage

[15, Proposition 2.4], which gave a sufficient condition for a space to be a D-space. Gruenhage introduced the following concept of smallness of sets with respect to neighbornets in [15]. Let U be a neighbornet of a space X. A set $A \subset X$ is *U-close* if $A \subset U\{x\}$ for every $x \in A$. We denote by $\mathscr{C}(U)$ the family of all U-close subsets of X. Note that $A \in \mathscr{C}(U)$ if, and only if, $A \times A \subset U$, in other words if, and only if, $y \in U\{x\}$ and $x \in U\{y\}$ for all $x, y \in A$. It follows that $St(x, \mathscr{C}(U)) = U\{x\} \cap U^{-1}\{x\}$ for every $x \in X$. Also note that $\mathscr{C}(U)$ is closed under monotone unions. As a consequence, every set in $\mathscr{C}(U)$ is contained in a maximal member of $\mathscr{C}(U)$.

We generalize the notion of a U-close set. Let U be a neighbornet of a space X. Note that C is U-close if, and only if, $\{x\}$ is a kernel of U in C for every $x \in C$. We say that a set $A \subset X$ is *countably U-close* if, for every relatively closed subset L of A, U has a kernel in L which is countable and closed discrete in X. We denote by $\mathscr{C}_\omega(U)$ the family of all countably U-close subsets of X.

Since the class of Lindelöf D-spaces is closed-hereditary, we see that closed Lindelöf D-subspaces are countably U-close for every U; in particular, σ-compact closed sets are countably U-close for every U. Whether closed Lindelöf subsets are countably U-close for all neighbornets in all spaces depends on the open problem whether all Lindelöf spaces are D.

We also want to generalize the notion of thickness. We say that a cover $\mathcal{L}$ of a space X is *almost thick* provided that we can assign $\mathcal{L}_H \in [\mathcal{L}]^{<\omega}$ and $L_H = \bigcup \mathcal{L}_H$ to each $H \in [X]^{<\omega}$ so that, for every non-closed $A \subset X$, there exists $H \in [A]^{<\omega}$ such that $L_H \cap (\overline{A} \setminus A) \neq \emptyset$. Like in [12, Lemma 2.1], we see that, in this definition, one could replace $\mathcal{L}_H \in \mathcal{L}^{<\omega}$ by $\mathcal{L}_H \in \mathcal{L}^{\leqslant\omega}$. As a consequence, $\mathcal{L}$ is almost thick provided that the family $\{\bigcup \mathcal{L}' \colon \mathcal{L}' \in [\mathcal{L}]^{\leqslant\omega}\}$ is almost thick.

The next result provides a sufficient condition for an open neighbornet to have a closed discrete kernel.

Proposition 3.1. *Let U be an open neighbornet of X such that the cover $\mathscr{C}_\omega(U)$ of X is almost thick. Then U has a closed discrete kernel in every closed subset of X.*

If we would replace "almost thick" by "thick" and "$\mathscr{C}_\omega(U)$" by "$\mathscr{C}(U)$" in Proposition 3.1, then the result would be a consequence of a result of Gruenhage [15, Proposition 2.4]. The following proof is an adaptation of Gruenhage's proof, which in turn was based on a proof by Buzyakova in [7]. Our proof also uses some ideas from [28, proof of Theorem 14].

If U satisfies the condition in Proposition 3.1, then for every closed $F \subset X$, the neighbornet V of the subspace F, defined by $V\{x\} = F \cap U\{x\}$, satisfies the same condition. As a consequence, to prove the proposition, we only need to show that U has a closed discrete kernel (in X).

Throughout the proof given below (including Lemma 3.2), we shall use the following notation and terminology. We assume that U is an open neighbornet of X, and the assignment $H \mapsto \mathscr{C}_H$ from $[X]^{<\omega}$ to $[\mathscr{C}_\omega(U)]^{<\omega}$ and the sets $C_H = \bigcup \mathscr{C}_H$ verify almost thickness of $\mathscr{C}_\omega(U)$. To make the proof less cumbersome, we introduce the following notation: for every $A \subset X$, we denote the set $\bigcup\{C_H\colon H \in [A]^{<\omega}\}$ by $\widehat{A}$. Without loss of generality, we can choose the assignment $H \mapsto \mathscr{C}_H$ so that we have $H \subset C_H$ for every H. Then we have $A \subset \widehat{A}$ for every $A \subset X$, and the condition for almost thickness can be stated as follows: *for each $A \subset X$, if A is closed in $\widehat{A}$, then A is closed in X*. Now let V be any open neighbornet of X such that $U \subset V$. Then $\mathscr{C}_\omega(U) \subset \mathscr{C}_\omega(V)$, and hence the assignment $H \mapsto \mathscr{C}_H$ also verifies almost thickness of $\mathscr{C}_\omega(V)$. We say that a set $L \subset X$ is *V-surrounded* if we have $\widehat{L} \subset V(L)$. Note that every kernel of V is V-surrounded. If $\mathcal{A}$ is a family of subsets of X, then $V(\bigcup \mathcal{A}) = \bigcup_{A \in \mathcal{A}} V(A)$ and if the family $\mathcal{A}$ is monotone, then $(\bigcup \mathcal{A})\widehat{\ } = \bigcup_{A \in \mathcal{A}} \widehat{A}$. As a consequence, a monotone union of V-surrounded sets is V-surrounded.

The following result is similar to [15, Lemma 2.4(b)].

Lemma 3.2. *Let V be a neighbornet of X with $U \subset V$, and let $\mathcal{L} = \{L_\alpha\colon \alpha < \lambda\}$ be a family of closed subsets of X such that, for each $\alpha < \lambda$, we have $L_\alpha \subset X \setminus V(\bigcup_{\beta<\alpha} L_\beta)$ and the set $\bigcup_{\beta \leqslant \alpha} L_\beta$ is V-surrounded. Then $\mathcal{L}$ is discrete.*

Proof. By Lemma 2.4, it suffices to show that the set $A_\alpha = \bigcup_{\beta<\alpha} L_\beta$ is closed for every $\alpha \leqslant \lambda$. Assume that this does not hold, and let γ be the least α such that $\overline{A_\alpha} \neq A_\alpha$. Note that γ is a limit ordinal. There exists $x \in \overline{A_\gamma} \setminus A_\gamma$ such that $x \in \widehat{A_\gamma}$. As the union of the monotone family $\{\bigcup_{\beta \leqslant \delta} L_\beta\colon \delta < \gamma\}$ of V-surrounded sets, the set A_γ is V-surrounded. It follows that $x \in \widehat{A_\gamma} \subset V(A_\gamma)$. Since $A_\gamma = \bigcup_{\beta<\gamma} L_\beta$, there exists $\beta < \gamma$ such that $x \in V(L_\beta)$. For every $\alpha > \beta$, we have that $L_\alpha \cap V(L_\beta) = \emptyset$. It follows that $x \notin \overline{\bigcup\{L_\alpha\colon \beta < \alpha < \gamma\}}$. Since $x \in \overline{A_\gamma}$ and $A_\gamma = A_{\beta+1} \cup \bigcup\{L_\alpha\colon \beta < \alpha < \gamma\}$, we have that $x \in \overline{A_{\beta+1}}$. Moreover, $x \in X \setminus A_\gamma \subset X \setminus A_{\beta+1}$. As a consequence, the set $A_{\beta+1}$ is not closed. However, we have that $\beta + 1 < \gamma$, and this contradicts the minimality of γ. $\square$

Proof of Proposition 3.1. We shall establish, by transfinite induction on $\omega \leqslant \kappa \leqslant |X|$, that X has the following property:

$\mathbf{P}_\kappa$: *For every open neighbornet $V \supset U$ and every $A \subset X$ with $|A| \leqslant \kappa$ and $V(A) \neq X$, there exists a non-empty closed discrete $E \subset X \setminus V(A)$ such that $|E| \leqslant \kappa$ and $A \cup E$ is V-surrounded.*

Once we have established $\mathbf{P}_\kappa$ for all $\omega \leqslant \kappa \leqslant |X|$, we can inductively define closed discrete sets D_α with $0 < |D_\alpha| \leqslant |\alpha|$ such that $D_\alpha \subset X \setminus U(\bigcup_{\beta<\alpha} D_\beta)$ and $\bigcup_{\beta \leqslant \alpha} D_\beta$ is U-surrounded. At some ordinal $\lambda < |X|^+$, we have that $U(\bigcup_{\alpha<\lambda} D_\alpha) = X$. The set $D = \bigcup_{\alpha<\lambda} D_\alpha$ is a kernel of U, and Lemma 3.2 shows that D is closed discrete.

Proof of $\mathbf{P}_\omega$. Suppose that $V \supset U$ is an open neighbornet of X and $A \subset X$ is countable with $V(A) \neq X$. Define $\{C_k^i\colon k < \omega\} \subset \mathscr{C}_\omega(V)$ and $D_i \subset X \setminus V(A)$ recursively for $i < N \leqslant \omega$ as follows. In the beginning, set $N = \omega$. Set $D_0 = \{x\}$ where $x \in X \setminus V(A)$. Enumerate the countable family $\bigcup\{\mathscr{C}_H\colon H \in [A \cup D_0]^{<\omega}\}$ as $\{C_k^0\colon k < \omega\}$. Assume that $\ell > 0$ and that, for each $j < \ell$, we have already chosen a countable closed discrete set $D_j \subset X \setminus V(A)$ and an enumeration $\{C_k^j\colon k < \omega\}$ of the family $\bigcup\{\mathscr{C}_H\colon H \in [A \cup \bigcup_{i \leqslant j} D_i]^{<\omega}\}$. If $\bigcup\{C_k^j\colon j < \ell \text{ and } k < \omega\} \subset V(A \cup \bigcup_{i<\ell} D_i)$, then we set $N = \ell$, we stop the recursion, and we note that the set $A \cup \bigcup_{i<\ell} D_i$ is V-surrounded and the set $\bigcup_{i<\ell} D_i$ is closed discrete; in this case the proof is complete. Otherwise, there exist $j < \ell$ and $k < \omega$ with $C_k^j \not\subset V(A \cup \bigcup_{i<\ell} D_i)$ and with $j + k$ as small as possible; since C_k^j is countably V-close, we can choose a countable closed discrete kernel D_ℓ for V in the set $C_k^j \setminus V(A \cup \bigcup_{i<\ell} D_i)$.

As noted above, the proof is complete if $N < \omega$. From now on, assume that $N = \omega$. We show that $A \cup \bigcup_{i<\omega} D_i$ is V-surrounded. Since $(A \cup \bigcup_{j<\omega} D_j)\widehat{\ } = \bigcup_{j<\omega}(A \cup \bigcup_{n \leqslant j} D_n)\widehat{\ } = \bigcup\{C_k^j\colon j, k < \omega\}$, it suffices to show that $C_k^j \subset V(A \cup \bigcup_{i<\omega} D_i)$ for all j and k. Let $j < \omega$ and $k < \omega$. There are at most $(j + k + 1)^2$ pairs (m, ℓ) with $m + \ell \leqslant j + k$, and this means that at some step $n \leqslant j + (j + k + 1)^2$ of the recursion, if $C_k^j \not\subset V(A \cup \bigcup_{i<n} D_i)$, then D_n is a kernel of V in $C_k^j \setminus V(A \cup \bigcup_{i<n} D_i)$ and therefore $C_k^j \subset V(A \cup \bigcup_{i \leqslant n} D_i)$. We have shown that $A \cup \bigcup_{i<\omega} D_i$ is V-surrounded.

To complete the proof of $\mathbf{P}_\omega$, we show that the set $D = \bigcup_{i<\omega} D_i$ is closed and discrete. Assume that D is not closed. Then there exists a point $x \in \widehat{D} \cap (\overline{D} \setminus D)$. We have $x \in (A \cup D)\widehat{\ }$ and it follows, since the set $A \cup D$ is V-surrounded, that $x \in V(A \cup D)$. It further follows, that there exists $j < \omega$ such that $x \in V(A \cup \bigcup_{i<j} D_i)$. The neighborhood $V(A \cup \bigcup_{i<j} D_i)$ of x is disjoint from D_m for every $m \geqslant j$. However, we now have a contradiction, because $x \in \overline{D} \setminus D$ and the set $\bigcup_{i<j} D_i$ is closed. We have shown that D is closed. To see that D is discrete, let $y \in D$. Then there exists $j < \omega$ such that $y \in D_j$. Now $V(D_j)$ is a neighborhood of y meeting at most $j + 1$ many of the closed discrete sets D_i, $i < \omega$. By the foregoing, the set $D = \bigcup_{i<\omega} D_i$ is discrete.

Proof of $\mathbf{P}_\kappa$, $\kappa > \omega$. Suppose that $\kappa > \omega$ and that we have proved $\mathbf{P}_{\kappa'}$ for each $\omega \leqslant \kappa' < \kappa$. Let $V \supset U$ be an open neighbornet and A a subset of X with $0 < |A| \leqslant \kappa$ and $V(A) \neq X$. Write $A = \{a_\beta\colon \beta < \kappa\}$. Define a neighbornet $\tilde{V}$ by setting $\tilde{V}\{x\} = V(\{x\} \cup A)$ for every $x \in X$. Define closed discrete sets E_β, $\beta < \kappa$, inductively so that, for every β, we have $|E_\beta| \leqslant |\beta|$, $E_\beta \subset X \setminus \tilde{V}(\{a_\gamma\colon \gamma < \beta\} \cup \bigcup_{\gamma<\beta} E_\gamma)$, $E_\beta \neq \emptyset$ if $\tilde{V}(\{a_\gamma\colon \gamma < \beta\} \cup \bigcup_{\gamma<\beta} E_\gamma) \neq X$, and the set $\{a_\gamma\colon \gamma \leqslant \beta\} \cup \bigcup_{\gamma \leqslant \beta} E_\gamma$ is $\tilde{V}$-surrounded.

Since we have $\tilde{V}(A) \subset \tilde{V}\{x\}$ for every $x \in X$, also the sets $\bigcup_{\gamma \leqslant \beta} E_\gamma$ are $\tilde{V}$-surrounded. By Lemma 3.2, the family $\{E_\alpha\colon \alpha < \lambda\}$ is discrete. Since each E_γ is closed and discrete, it follows that the set $E = \bigcup\{E_\beta\colon \omega \leqslant \beta < \kappa\}$ is closed and discrete. Note that $|E| \leqslant \kappa$. As a monotone union of $\tilde{V}$-surrounded sets, the set $A \cup E$ is $\tilde{V}$-surrounded. Since $V(A \cup E) = \tilde{V}(A \cup E)$, the set $A \cup E$ is also V-surrounded. This completes the proof of $\mathbf{P}_\kappa$, and the proof of Proposition 3.1. □

Since point-countably expandable covers are thick, we have the following consequence of Proposition 3.1.

Corollary 3.3. *Let U be an open neighbornet of X such that there exists a point-countably expandable cover of X by countably U-close sets. Then U has a closed discrete kernel.*

Proposition 3.1 provides us with a sufficient condition for a space to be a D-space. We record here some special cases where that condition is applicable.

Since closed Lindelöf D-subsets are countably U-close for any U, we have the following result.

Corollary 3.4. *If X has an almost thick cover by closed Lindelöf D-subsets, then X is a D-space.*

Peng has shown that X is a D-space provided that X has a closure-preserving cover by closed D-subsets [23, Theorem 13]. Since closure-preserving closed covers are thick, we are led to ask whether the above result would remain valid with “Lindelöf” omitted?

Note that, for every neighbornet V of X, the cover $\{V^{-1}\{x\}\colon x \in X\}$ is thick. Consequently, the following result obtains.

Corollary 3.5. *Assume that X has a neighbornet V such that $\overline{V^{-1}\{x\}}$ is a Lindelöf D-space for every $x \in X$. Then X is a D-space.*

Next we show that Proposition 3.1 is also useful in connection with some generalizations of D-spaces.

Several of the results below deal with thick partitions. Note that if $\mathcal{L}$ is a point-countably expandable cover of X and we write $\mathcal{L} = \{L_\alpha\colon \alpha < \kappa\}$, then $\mathcal{L}$ has a refinement $\{L_\alpha \setminus \bigcup_{\beta<\alpha} L_\beta\colon \alpha < \kappa\}$ which is a point-countably expandable, and hence thick, partition of X. We do not know whether every thick cover is refined by a thick partition, but we shall indicate various thick partitions below.

A partition $\mathcal{N}$ of a space X is *scattered* provided that we can write $\mathcal{N} = \{N_\alpha\colon \alpha < \lambda\}$ for some ordinal λ so that, for every $\beta < \lambda$, the set $\bigcup_{\alpha<\beta} N_\alpha$ is open in X [30].

Several hereditary covering properties have been characterized by properties of scattered partitions. For example, X is hereditarily metacompact (hereditarily metaLindelöf) if, and only if, every scattered partition of X is point-finitely expandable (point-countably expandable) (see [22, Theorem 6.3] and [12, proof of Lemma 3.17]). We shall now prove a similar characterization for hereditary thick coveredness.

Proposition 3.6. *A space is hereditarily thickly covered if, and only if, every scattered partition of the space is thick.*

Proof. *Sufficiency.* Assume that every scattered partition of X is thick. To show that X is hereditarily thickly covered, it suffices to show that every family $\mathcal{G}$ of open subsets of X is thick in the subspace $\bigcup\mathcal{G}$ of X. Let $\mathcal{G}=\{G_\alpha\colon \alpha<\kappa\}$ consist of open subsets of X. Set $G_\kappa=X$, and let $S_\alpha=G_\alpha\setminus\bigcup_{\beta<\alpha}G_\beta$ for every $\alpha\leqslant\kappa$. Then $\mathcal{S}=\{S_\alpha\colon \alpha\leqslant\kappa\}$ is a scattered partition of X. As a consequence, $\mathcal{S}$ is thick in X. It follows that the family $\mathcal{S}'=\{S_\alpha\colon \alpha<\kappa\}$ is thick in the subspace $\bigcup\mathcal{S}'=\bigcup\mathcal{G}$. Moreover, $\mathcal{S}'$ refines $\mathcal{G}$, and hence $\mathcal{G}$ is thick in $\bigcup\mathcal{G}$.

Necessity. Assume that X is hereditarily thickly covered. We use transfinite induction on the ordinal μ to show that, for every monotone family $\mathcal{G}=\{G_\alpha\colon \alpha<\mu\}$ of open subsets of X, the family $\{G_\alpha\setminus\bigcup_{\beta<\alpha}G_\beta\colon \alpha<\mu\}$ is a thick partition of the subspace $\bigcup\mathcal{G}$ of X.

Since every countable cover is thick, the result holds when μ is countable. Let λ be an ordinal such that the result holds for every $\mu<\lambda$. To prove the result for λ, let $\mathcal{G}=\{G_\alpha\colon \alpha<\lambda\}$ be a monotone family of open subsets of X. Denote by Z the subspace $\bigcup\mathcal{G}$ of X. The open cover $\mathcal{G}$ of Z is thick. It follows, since $\mathcal{G}$ is a monotone family, that we can assign $\alpha_H<\lambda$ for each $H\in[Z]^{<\omega}$ so that we have $\overline{A}\cap Z\subset\bigcup\{G_{\alpha_H}\colon H\in[A]^{<\omega}\}$ for every $A\subset Z$. Without loss of generality, we can assume that $\alpha_J\leqslant\alpha_H$ whenever $J\subset H$.

For every $\alpha<\lambda$, set $S_\alpha=G_\alpha\setminus\bigcup_{\beta<\alpha}G_\beta$. Let $\mu<\lambda$. It follows from our inductive assumption that the family $\{S_\alpha\colon \alpha<\mu\}$ is a thick partition of the open subspace $Z_\mu=\bigcup_{\alpha<\mu}G_\alpha$ of X. As a consequence, we can assign $E^\mu_H\in[\mu]^{<\omega}$ and $S^\mu_H=\bigcup_{\alpha\in E^\mu_H}S_\alpha$ to each $H\in[Z_\mu]^{<\omega}$ so that we have $\overline{A}\cap Z_\mu\subset\bigcup\{S^\mu_H\colon H\in[A]^{<\omega}\}$ for every $A\subset Z_\mu$. For every $H\in[Z]^{<\omega}\setminus[Z_\mu]^{<\omega}$, set $E^\mu_H=S^\mu_H=\emptyset$.

For every $H\in[Z]^{<\omega}$, let $E_H=\{\alpha_H\}\cup\bigcup\{E^{\alpha_L}_J\colon J\subset H \text{ and } L\subset H\}$ and $S_H=\bigcup_{\alpha\in E_H}S_\alpha$. To complete the proof, let $A\subset Z$. We show that $\overline{A}\cap Z\subset\bigcup\{S_H\colon H\in[A]^{<\omega}\}$. Let $x\in\overline{A}\cap Z$. There exists $H\in[A]^{<\omega}$ such that $x\in G_{\alpha_H}$. Denote the ordinal α_H by μ. If $x\in S_\mu$, then $x\in S_H$. Otherwise, $x\in Z_\mu$ and it follows, since Z_μ is open, that $x\in\overline{A\cap Z_\mu}$. As a consequence, there exists $J\in[A\cap Z_\mu]^{<\omega}$ such that $x\in S^\mu_J$. Let $K=H\cup J$. Then $K\in[A]^{<\omega}$ and $x\in S^\mu_J=S^{\alpha_H}_J\subset S_K$. This completes the proof of thickness of the partition $\{S_\alpha\colon \alpha<\lambda\}$ of Z. □

Let us call a space *almost thickly covered* if every open cover of the space is almost thick. Similarly as in the beginning of the preceding proof, we see that a space is hereditarily almost thickly covered if every scattered partition of the space is almost thick. However, we do not know whether the converse holds.

Problem 3.7. Is every scattered partition of a hereditarily almost thickly covered space almost thick?

With the help of Propositions 3.1 and 3.6, we can show that hereditarily thickly covered spaces satisfy certain weaker versions of the D-property.

A space X is aD provided that, for every open cover $\mathcal{V}$ of X, there exists a closed discrete set $D\subset X$, and for every $x\in D$, a set $V_x\in(\mathcal{V})_x$ such that $\{V_x\colon x\in D\}$ covers X [5]. The space X is *linearly D*, provided that for every monotone open cover $\mathcal{U}$ of X, if $X\notin\mathcal{U}$, then there exists a closed discrete set $D\subset X$ such that D is not contained in any member of $\mathcal{U}$ (see [17]).

Proposition 3.8. *A hereditarily thickly covered space is aD and linearly D.*

Proof. Let X be a hereditarily thickly covered space. We show first that X is aD. Let $\mathcal{V}$ be an open cover of X. Write $\mathcal{V}=\{V_\alpha\colon \alpha<\lambda\}$ for some ordinal λ. For every $\alpha<\lambda$, let $J_\alpha=V_\alpha\setminus\bigcup_{\beta<\alpha}V_\beta$. The family $\mathcal{J}=\{J_\alpha\colon \alpha<\lambda\}$ is a scattered partition of X. By Proposition 3.6, $\mathcal{J}$ is thick. For every $x\in X$, let β_x be the smallest ordinal in the set $\{\alpha<\lambda\colon x\in V_\alpha\}$. Define a neighbornet U of X by the condition $U\{x\}=V_{\beta_x}$. For every $\alpha<\lambda$, we have that $J_\alpha=\{x\in X\colon U\{x\}=V_\alpha\}$. It follows that $\mathcal{J}$ consists of U-close sets. By Proposition 3.1, the neighbornet U has a closed discrete kernel D. This set D verifies that X satisfies the condition for aD with respect to the cover $\mathcal{V}$.

Next we show that X is linearly D. Let $\mathcal{U}$ be a monotone open cover of X such that $X\notin\mathcal{U}$. Then $\mathcal{U}$ has a subcover $\mathcal{V}=\{V_\alpha\colon \alpha<\lambda\}$ such that $V_\alpha\subsetneq V_\beta$ whenever $\alpha<\beta<\lambda$. Define $\mathcal{J}$, U and D as in the first part of the proof, and note that D is not contained in any member of $\mathcal{V}$ and hence not in any member of $\mathcal{U}$. □

Problem 3.9. Does the above result remain valid for hereditarily almost thickly covered spaces?

Note that a positive solution to Problem 3.7 would yield a positive solution to Problem 3.9.

4. *t*-Metrizable spaces and predictable networks

In this section, we use the results above to prove that every t-metrizable space is a D-space. A consequence of this is that many interesting spaces are D-spaces. We start by exhibiting some important subclasses of the class of t-metrizable spaces. We need the following auxiliary result.

Lemma 4.1. *Let $\mathcal{L}$ be a discrete family of subsets of a thickly covered space X. Then the partition $\mathcal{L} \cup \{X \setminus \bigcup \mathcal{L}\}$ of X is thick.*

Proof. The result holds trivially if $|\mathcal{L}| \leqslant 1$. Assume that $|\mathcal{L}| > 1$. For every $L \in \mathcal{L}$, let $G(L) = X \setminus \overline{\bigcup(\mathcal{L} \setminus \{L\})}$, and note that $G(L)$ is an open set containing L. The family $\mathcal{G} = \{G(L)\colon\ L \in \mathcal{L}\}$ is an open cover of X, and it follows, since X is thickly covered, that $\mathcal{G}$ is thick. Hence we can assign $\mathcal{L}_H \in [\mathcal{L}]^{<\omega}$ and $G_H = \bigcup\{G(L)\colon\ L \in \mathcal{L}_H\}$ to each $H \in [X]^{<\omega}$ so that we have $\overline{A} \subset \bigcup\{G_H\colon\ H \in [A]^{<\omega}\}$ for every $A \subset X$.

Let $C = X \setminus \bigcup \mathcal{L}$ and $\mathcal{L}' = \mathcal{L} \cup \{C\}$. For every $H \in [X]^{<\omega}$, set $\mathcal{L}'_H = \mathcal{L}_H \cup \{C\}$ and $L'_H = \bigcup \mathcal{L}'_H$. We show that the assignment $H \mapsto \mathcal{L}'_H$ verifies thickness of the cover $\mathcal{L}'$. Let $A \subset X$ and $x \in \overline{A}$. If $x \in C$, then $x \in \bigcup\{L'_H\colon\ H \in [A]^{<\omega}\}$. Assume that $x \in \bigcup \mathcal{L}$. Since $x \in \overline{A}$, there exists $H \in [A]^{<\omega}$ and $L \in \mathcal{L}_H$ such that $x \in G(L)$. However, since $x \in \bigcup \mathcal{L}$, it follows from $x \in G(L)$ that $x \in L$. As a consequence, we have $x \in L'_H$. $\square$

Note that if X is hereditarily thickly covered, then the conclusion of the above lemma holds for every *relatively* discrete family $\mathcal{L}$ in X. This follows easily from the lemma, because there exists an open $G \subset X$ such that $\bigcup \mathcal{L} \subset G$ and $\mathcal{L}$ is a discrete family in the subspace G.

Recall that X is a *σ-space* if X has a σ-discrete closed network, and X is a *primitive σ-space* if X has a network which is the union of countably many scattered partitions.

For a topological vector space L, we denote by L_w the space obtained when L is equipped with its weak topology. For a topological space Y, we denote by $C_p(Y)$ the set $C(Y)$ consisting of all continuous real-valued functions on Y equipped with the topology of pointwise convergence.

Proposition 4.2. *The following kinds of spaces are t-metrizable*:

(a) *Spaces L_w, where L is a metrizable locally convex topological vector space.*
(b) *Spaces $C_p(K)$, where K is a compact Hausdorff space.*
(c) *Spaces with a point-countably expandable network.*
(d) *Thickly covered σ-spaces.*
(e) *Hereditarily thickly covered primitive σ-spaces.*

Proof. For (a) and (b), see [12, Theorem 3.2]. (c) follows from [12, Proposition 3.8 and Theorem 3.10]. If X satisfies (d) (or (e)), then it follows from Lemma 4.1 (or Proposition 3.6) that X has a network which is the union of countably many thick partitions; by [12, Theorem 3.4], X is t-metrizable. $\square$

Since stratifiable spaces are paracompact σ-spaces (see [9] and [19]), it follows from (d) that every stratifiable space is t-metrizable.

To prove that t-metrizable spaces are D-spaces, it is useful to introduce a network condition satisfied by all t-metrizable spaces. We shall consider networks $\mathcal{N}$ for which a subset A can "predict", by finitary conditions, which members of $\mathcal{N}$ meet the closure of A.

Definition 4.3. A family $\mathcal{L}$ of subsets of a space X is *predictable* if there exists an assignment $H \mapsto \mathcal{L}_H$ from $[X]^{<\omega}$ to $[\mathcal{L}]^{<\omega}$ such that we have $(\mathcal{L})_{\overline{A}} \subset \bigcup\{\mathcal{L}_H\colon\ H \in [A]^{<\omega}\}$ for every infinite $A \subset X$.

We record some straightforward consequences of the definition.

Remarks.

(i) It suffices above that the assignment $H \mapsto \mathcal{L}_H$ is from $[X]^{<\omega}$ to $[\mathcal{L}]^{\leqslant\omega}$ (compare with [12, proof of Lemma 2.1]).
(ii) As a consequence of (i), every "σ-predictable" family is predictable.
(iii) Every point-countably expandable family is predictable and every predictable family is point-countable.
(iv) Every predictable cover is thick and every thick partition is predictable.
(v) If $\mathcal{L}$ is predictable and $K_L \subset L$ for every $L \in \mathcal{L}$, then the family $\{K_L\colon\ L \in \mathcal{L}\}$ is predictable.
(vi) As a consequence of (v), every predictable cover of X is refined by a predictable partition of X.

Note that, by Remark (iii), an open family $\mathcal{U}$ is predictable if, and only if, $\mathcal{U}$ is point-countable.

The analogues of Remarks (v) and (vi) hold for point-countable expandability: if $\mathcal{L}$ is point-countably expandable and $K_L \subset L$ for every $L \in \mathcal{L}$, then the family $\{K_L\colon\ L \in \mathcal{L}\}$ is point-countably expandable. It follows that every point-countably expandable cover is refined by a point-countably expandable partition.

The next result is a consequence of Remarks (vi) and (iv).

Lemma 4.4. *The following are equivalent for a cover $\mathcal{H}$ of X:*

A. *$\mathcal{H}$ has a predictable refinement.*
B. *$\mathcal{H}$ is refined by a predictable partition.*
C. *$\mathcal{H}$ is refined by a thick partition.*

We do not know whether the above conditions are equivalent to thickness of $\mathcal{H}$. In other words, we do not know whether every thick cover is refined by a thick partition. A related problem deals with the covering property associated with predictable families.

Problem 4.5. Does every open cover of a thickly covered space have a predictable refinement?

Note that, by Proposition 3.6 and Lemma 4.4, the above problem has a positive solution for *hereditarily* thickly covered spaces.

By [12, Theorem 3.4], every t-metrizable space has a network which is the union of countably many thick partitions. As a consequence, Remarks (iv) and (ii) above yield the following result.

Proposition 4.6. *Every t-metrizable space has a predictable network.*

Let X be a space with a predictable network. Then every subspace of X has a predictable network, and hence X is hereditarily thickly covered. It follows from Proposition 3.6 that every scattered partition of X is thick, but the next result gives a stronger conclusion.

Proposition 4.7. *Suppose that X has a predictable network and U is a neighbornet of X. Then X has a thick partition consisting of U-close sets.*

Proof. Let $\mathcal{N}$ be a predictable network of X. For every $N \in \mathcal{N}$, let $\widetilde{N} = \{x \in N\colon\ N \subset U\{x\}\}$, and note that $\widetilde{N}$ is U-close. Since $\mathcal{L}$ is a network of X, the family $\widetilde{\mathcal{N}} = \{\widetilde{N}\colon\ N \in \mathcal{N}\}$ covers X. By Remark (v) following Definition 4.3, the cover $\widetilde{\mathcal{N}}$ is predictable, and by Remark (vi), the cover $\widetilde{\mathcal{N}}$ is refined by a predictable partition. $\square$

The following is a consequence of Propositions 3.1 and 4.7.

Theorem 4.8. *Every space with a predictable network is a D-space.*

Proposition 4.6 and Theorem 4.8 have the following consequence.

Corollary 4.9. *Every t-metrizable space is a D-space.*

Since t-metrizability is a hereditary property, it follows from Corollary 4.9 that any space satisfying one of the conditions of Proposition 4.2 is hereditarily D. That $C_p(K)$-spaces are hereditarily D was originally established by Buzyakova in [7]. The result that a space with a point-countably expandable network is a D-space generalizes the result of Arhangel'skii and Buzyakova that a space with a point-countable base is a D-space [5, Theorem 2].

To close this section, we consider two strong monolithicity properties which generalize the existence of a predictable network.

The concept of "monotone monolithicity" was defined by V. Tkachuk in [29]. A family $\mathcal{M}$ of subsets of X is said to be a *network at a point* $x \in X$ provided that every neighborhood of x contains some set of the family $(\mathcal{M})_x$. *Monotonically monolithic* spaces can be characterized as those spaces X for which one can associate a countably family $\mathcal{N}_H$ of subsets of X with each finite subset H of X in such a way that, for every $A \subset X$, the family $\bigcup_{H \in [A]^{<\omega}} \mathcal{N}_H$ is a network at every point of the set $\overline{A}$. Note that we can choose the families $\mathcal{N}_H$ so that $\{x\} \in \mathcal{N}_H$ for every $x \in H$, and hence it is enough above to require that the family $\bigcup_{H \in [A]^{<\omega}} \mathcal{N}_H$ is a network at every point of the set $\overline{A} \setminus A$.

Proposition 4.10. *If X has a predictable network, then X is monotonically monolithic.*

Proof. Let $\mathcal{N}$ be a predictable network for X. Let the assignment $H \mapsto \mathcal{N}_H$ from $[X]^{<\omega}$ to $[\mathcal{N}]^{<\omega}$ verify predictability of $\mathcal{N}$. For every $A \subset X$, we have that $(\mathcal{N})_{\overline{A}} \subset \bigcup\{\mathcal{N}_H\colon\ H \in [A]^{<\omega}\}$ and it follows, since $\mathcal{N}$ is a network of X, that the family $\bigcup\{\mathcal{N}_H\colon\ H \in [A]^{<\omega}\}$ is a network at every point of $\overline{A}$. □

We do not know whether the converse of the above result holds.

Problem 4.11. Does every monotonically monolithic space have a predictable network?

According to [29, Theorem 2.14], every monotonically monolithic space is a D-space; in light of Proposition 4.10, this result generalizes Theorem 4.8. On the other hand, we can derive the D-space property of monotonically monolithic spaces from Proposition 3.1 and the following observation.

Proposition 4.12. *Let U be a neighbornet of a monotonically monolithic space X. Then the cover $\mathscr{C}(U)$ of X is thick.*

Proof. Let the families $\mathcal{N}_H$, $H \in [X]^{<\omega}$ verify the monotonic monolithicity of X. Denote by $\mathcal{N}$ the network $\bigcup\{\mathcal{N}_H\colon\ H \in [X]^{<\omega}\}$ of X. For every $E \subset X$, let $\widetilde{E} = \{x \in E\colon\ E \subset U\{x\}\}$, and note that $\widetilde{E} \in \mathscr{C}(U)$. For every $\mathcal{E} \subset \mathcal{P}(X)$, let $\widetilde{\mathcal{E}} = \{\widetilde{E}\colon\ E \in \mathcal{E}\}$. Since $\mathcal{N}$ is a network of X, the family $\widetilde{\mathcal{N}}$ covers X. We show that the cover $\widetilde{\mathcal{N}}$ is thick. Let $A \subset X$ and $x \in \overline{A}$. The family $\bigcup_{H\in[A]^{<\omega}} \mathcal{N}_H$ is a network at x, and hence there exists $H \in [A]^{<\omega}$ and $N \in \mathcal{N}_H$ such that $x \in N \subset U\{x\}$. Now $x \in \widetilde{N} \in \widetilde{\mathcal{N}}_H$. The foregoing and the remark following Definition 2.3 show that the assignment $H \mapsto \widetilde{\mathcal{N}}_H$ verifies thickness of the cover $\widetilde{\mathcal{N}}$. Since $\widetilde{\mathcal{N}} \subset \mathscr{C}(U)$, also the cover $\mathscr{C}(U)$ is thick. □

Corollary 4.13. *Every monotonically monolithic space is hereditarily thickly covered.*

Propositions 3.1 and 4.12 have the following consequence.

Corollary 4.14. *([29]) Every monotonically monolithic space is a D-space.*

With the help of Propositions 4.10 and 4.12 and some results from [12] and [29], we can answer five of the ten questions listed at the end of Tkachuk's paper [29].

[29, Question 3.2] asks whether every stratifiable space is monotonically monolithic. However, as noted after Proposition 4.2, stratifiable spaces are t-metrizable. As a consequence, Propositions 4.6 and 4.10 show that the answer to [29, Question 3.2] is "yes". Since M_1-spaces are stratifiable, also the answer to [29, Question 3.1] is "yes".

[29, Question 3.6] asks whether every monotonically monolithic compact (Hausdorff) space is Corson compact. To answer this question, let X be a monotonically monolithic compact Hausdorff space. By [29, Proposition 2.3(iv) and Theorem 2.10], the space $X \times X$ is hereditarily monotonically monolithic and hence, by Corollary 4.13, hereditarily thickly covered. By [12, Theorem 2.12], X is Corson compact. As a consequence, the answer to [29, Question 3.6] is "yes". It follows, by well-known properties of Corson compact spaces (see [1,2,16]), that also the answers to [29, Questions 3.7 and 3.8] are "yes".

Recently, Peng introduced the class of "weakly monotonically monolithic" spaces, which generalize Tkachuk's monotonically monolithic spaces. Peng generalized the result of Corollary 4.14 above by showing that weakly monotonically monolithic spaces are D-spaces.

Weakly monotonically monolithic spaces can be characterized as those spaces X for which one can associate a countably family $\mathcal{N}_H$ of subsets of X with each finite subset H of X in such a way that, for every non-closed $A \subset X$, the family $\mathcal{N}(A) = \bigcup_{H\in[A]^{<\omega}} \mathcal{N}_H$ is a network at some point of the set $\overline{A} \setminus A$.

The following consequence of Proposition 3.1 shows that all weakly monotonically monolithic spaces are D-spaces.

Lemma 4.15. *A neighbornet U of X has a closed discrete kernel provided that one can associate $\mathcal{N}_H \in [\mathcal{P}(X)]^{\leqslant\omega}$ with each $H \in [X]^{<\omega}$ in such a way that, for every non-closed subset A of X, there exist $x \in \overline{A} \setminus A$, $H \in [A]^{<\omega}$ and $N \in \mathcal{N}_H$ such that $x \in N \subset U\{x\}$.*

Proof. By Proposition 3.1, it suffices to show that the family $\mathscr{C}(U)$ is almost thick. For every $E \subset X$, let $\widetilde{E} = \{x \in E\colon\ E \subset U\{x\}\}$, and note that $\widetilde{E} \in \mathscr{C}(U)$. For every $H \in [X]^{<\omega}$, let $C_H = \bigcup\{\widetilde{N}\colon\ N \in \mathcal{N}_H\}$, and note that C_H is the union of countably many U-close sets. By the remark made after the definition of almost thickness, we can use the assignment $H \mapsto C_H$ to verify almost thickness of $\mathscr{C}(U)$. Let $A \subset X$ be non-closed. Then there exist $x \in \overline{A} \setminus A$, $H \in [A]^{<\omega}$ and $N \in \mathcal{N}_H$ such that $x \in N \subset U\{x\}$. Now $x \in \widetilde{N} \subset C_H$. □

Corollary 4.16. *([28]) Every weakly monotonically monolithic space is a D-space.*

5. On unions of *D*-spaces

Arhangel'skii and Buzyakova proved in [5, Theorem 5] that a regular space is a D-space if the space is the union of finitely many subspaces with σ-disjoint bases. Arhangel'skii extended that result in [3, Theorem 1.14] by showing that a regular space is a D-space provided that the space is the union of finitely many subspaces with point-countable bases. We do not know whether "point-countable base" can be replaced by "point-countably expandable network":

Problem 5.1. Is a regular space a D-space if the space is the union of finitely many subspaces with point-countably expandable networks?

We can obtain a partial solution to the above problem by strengthening "point-countable".

Definition 5.2. A family $\mathcal{L}$ of sets is *strongly point-countable* if every centered subfamily of $\mathcal{L}$ is countable.

Note that every family of finite order is strongly point-countable. Moreover, every σ-strongly point-countable family is strongly point-countable; in particular, every σ-disjoint family is strongly point-countable.

Lemma 5.3. *If $A \subset X$ and $\mathcal{G}$ is a strongly point-countable family of open sets in A, then the family $\mathcal{G}$ can be extended to a strongly point-countable family of open sets in $\overline{A}$.*

Proof. For every $G \in \mathcal{G}$, let $V(G)$ be an open subset of $\overline{A}$ such that $V(G) \cap A = G$. It is not difficult to check that the family $\{V(G)\colon\ G \in \mathcal{G}\}$ is a strongly point-countable open family of $\overline{A}$. $\square$

Lemma 5.4. *Suppose that $X = A \cup B$ where the subspaces A and B have strongly point-countably expandable networks. Then the subspace $\overline{A} \cap \overline{B}$ has a strongly point-countably expandable network.*

Proof. Let $\mathcal{H}$ be a strongly point-countably expandable network for A and $\mathcal{J}$ a strongly point-countably expandable network for B. Since $A \cup B = X$, the family $\mathcal{H} \cup \mathcal{J}$ is a network of X.

By Lemma 5.3, $\mathcal{H}$ is strongly point-countably expandable in $\overline{A}$ and $\mathcal{J}$ is strongly point-countably expandable in $\overline{B}$. It follows that the family $\mathcal{K} = \{L \cap \overline{A} \cap \overline{B}\colon\ L \in \mathcal{H} \cup \mathcal{J}\}$ is strongly point-countably expandable in $\overline{A} \cap \overline{B}$. Moreover, $\mathcal{K}$ is a network of $\overline{A} \cap \overline{B}$. $\square$

Proposition 5.5. *Suppose that X is the union of finitely many subspaces with strongly point-countably expandable networks. Then X is a D-space.*

Proof. Use Proposition 4.2(c) and Lemma 5.4, and substitute "strongly point-countably expandable network" for each occurrence of "σ-disjoint base" in [5, proof of Theorem 5]. $\square$

Corollary 5.6. *If a space is the union of finitely many screenable σ-spaces, then it is a D-space.*

Proof. The stated result follows from Proposition 5.5 once we show that a screenable σ-space has a σ-disjointly expandable network. This in turn follows when we observe that discrete families in screenable spaces have σ-disjointly expandable refinements.

Suppose that X is screenable and $\mathcal{L}$ is a discrete family of subsets of X. Then X has an open cover $\mathcal{G}$ such that $|(\mathcal{L})_G| \leqslant 1$ for every $G \in \mathcal{G}$. Let $\mathcal{U}$ be a σ-disjoint open refinement of $\mathcal{G}$. The family $\{U \cap L\colon\ U \in \mathcal{U}$ and $L \in \mathcal{L}\}$ is a σ-disjointly expandable (and σ-discrete) refinement of $\mathcal{L}$. $\square$

It is shown in [6] that a space is a D-space provided that the space is the union of countably many closed D-subspaces. As a generalization, we show that a space is a D-space provided that the space has a "linearly closure-preserving" cover consisting of closed D-subspaces.

Proposition 5.7. *Suppose that $X = \bigcup\{X_\alpha\colon\ \alpha < \lambda\}$, where each X_α is a D-subspace and for each $\beta < \lambda$, the set $\bigcup_{\alpha<\beta} X_\alpha$ is closed in X. Then X is a D-space.*

Proof. We use transfinite induction to define closed discrete sets D_β such that D_β is a kernel of V in the set $X_\beta \setminus V(\bigcup_{\alpha<\beta} D_\alpha)$, for every $\beta < \lambda$. Since X_0 is a D-space and X_0 is closed in X, there exists a closed discrete kernel D_0 of V in X_0.

Let $\beta < \lambda$ be such that D_α has been defined for each $\alpha < \beta$. The closed subspace $X_\beta \setminus V(\bigcup_{\alpha<\beta} D_\alpha)$ of the D-space X_β is a D-space. Hence there exists a closed discrete kernel D_β of U in $X_\beta \setminus V(\bigcup_{\alpha<\beta} D_\alpha)$. Note that, since $X_\beta \setminus V(\bigcup_{\alpha<\beta} D_\alpha) = \bigcup_{\alpha\leqslant\beta} X_\alpha \setminus V(\bigcup_{\alpha<\beta} D_\alpha)$, the set D_β is closed in X.

We have that $\bigcup_{\alpha<\beta} X_\alpha \subset V(\bigcup_{\alpha<\beta} D_\alpha)$ for every $\beta < \lambda$. It follows that the set $\bigcup_{\alpha<\lambda} D_\alpha$ is a kernel of V. Note that, for every $\beta \leqslant \lambda$, we have $\overline{\bigcup_{\alpha<\beta} D_\alpha} \subset \overline{\bigcup_{\alpha<\beta} X_\alpha} = \bigcup_{\alpha<\beta} X_\alpha \subset V(\bigcup_{\alpha<\beta} D_\alpha)$. Since $D_\alpha \subset X \setminus V(\bigcup_{\beta<\alpha} D_\beta)$ for every $\alpha < \lambda$, Lemma 2.4 shows that the family $\{D_\alpha\colon\ \alpha < \lambda\}$ is discrete. It follows, by closed discreteness of the sets D_α, that the set $\bigcup_{\alpha<\lambda} D_\alpha$ is closed discrete. □

Since the union of countably many closed D-subspaces is D, we have the following consequence of Proposition 5.7.

Corollary 5.8. *([23]) Suppose that X has a σ-closure-preserving cover consisting of closed D-subspaces. Then X is a D-space.*

The ordinal space ω_1 and the locally compact space Γ from [11] show that, in general, "local" does not imply "global" for the D-property. However, Peng showed in [24] that every submetacompact locally D-space is a D-space. Peng derived his theorem as a corollary to a result dealing with topological games. We shall give an alternative proof for Peng's theorem.

To prove the theorem, we need an auxiliary result. Let $\mathcal{L}$ be a family of sets. We say that a set A is *$\mathcal{L}$-small* if A is contained in some member of $\mathcal{L}$.

Lemma 5.9. *A space X is a D-space provided that X has a finite open cover $\mathcal{G}$ such that every $\mathcal{G}$-small closed subspace of X is a D-space.*

Proof. We use induction on the size of the finite open cover. Assume that the result holds for open covers of size less than n. To prove the result for open covers of size n, let X be a space with an open cover $\mathcal{G}$ of size n such that every $\mathcal{G}$-small closed subspace of X is a D-space. We show that X is a D-space. Choose a member H of $\mathcal{G}$, and let $\mathcal{G}' = \mathcal{G} \setminus \{H\}$. The closed subset $X \setminus \bigcup \mathcal{G}'$ of X is contained in H and hence $X \setminus \bigcup \mathcal{G}'$ is a D-space. Now let U be an open neighbornet of X. There exists a closed and discrete kernel D of U in $X \setminus \bigcup \mathcal{G}'$. Let $Z = X \setminus UD$, and note that $Z \subset \bigcup \mathcal{G}'$. Since Z is closed in X, every $\mathcal{G}'$-small closed subspace of Z is a D-space. By the inductive assumption, Z is a D-space. Hence there exists a closed and discrete kernel E of U in Z. The set $D \cup E$ is a closed discrete kernel of U in X. □

Theorem 5.10. *([24]) Every submetacompact locally D-space is a D-space.*

Proof. Let X be a submetacompact locally D-space. Let $\mathcal{D} = \{A \subset X\colon\ A$ is a D-subspace of $X\}$. Since X is locally D, the open family $\mathcal{U} = \{\operatorname{int} A\colon\ A \in \mathcal{D}\}$ covers X. By [21, Theorem 4.4], X has a σ-closure-preserving closed cover $\mathcal{F}$ such that, for every $F \in \mathcal{F}$, there exists a finite $\mathcal{U}_F \subset \mathcal{U}$ with $F \subset \bigcup \mathcal{U}_F$.

To show that X is a D-space, it suffices, by Corollary 5.8, to show that every member of $\mathcal{F}$ is a D-space. Let $F \in \mathcal{F}$. Every $\mathcal{U}_F$-small closed subset of F is $\mathcal{D}$-small and closed in X and hence it is a D-space. By Lemma 5.9, F is a D-space. □

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